Initiation of Roll Waves

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The theory proposed by Jeffreys to explain roll-wave transition on a liquid surface is applied to the concurrent flow of a gas and liquid. Data are presented for the concurrent flow of air with water-glycerine solutions, water-butanol solutions, and water-sodium lauryl sulfate solutions. Agreement is obtained between theory and experiment.

Waves moving over a stationary deep liquid cause particles in the bulk fluid to move in circular paths. However if the fluid is shallow compared with the wave length, fluid particles are disturbed much more in the direction of propagation of the wave than perpendicular to this direction. For this case a shallow water approximation holds; pressure variation in a direction perpendicular to the wave propagation may be assumed to occur only because of variation of the hydrostatic head (*36*). For a flowing liquid similar long wave length disturbances occur which might be thought of as surges in the volumetric flow of the liquid. From a consideration of the law of conservation of mass (*28*) it can be shown that the crests of long wave length disturbances on a flowing liquid move faster than the troughs. This will cause the downstream end of the wave to steepen and to roll over upon itself. Such waves are called *roll waves*. They were first described by Cornish, who observed them in water runways in the mountains (*8*). Photographs of them are given by Cornish in his book. These waves which occur in inclined liquid flows are important in drainage problems and have received extensive treatment in the literature (*9, 11, 19, 24, 34, 35, 36, 37*). Roll waves have also been noted in wetted wall towers (*14, 20*), where they can increase the rates of mass transfer and make the interpretation of data obtained in these systems more difficult. Waves similar in appearance to the roll waves obtained in open channel flows have also been reported in the concurrent flow of a gas and a liquid film (*15, 18, 25, 26, 33*). These should be of interest in obtaining an understanding of some of the phenomena occurring in two-phase transport in a pipeline. They are also of interest in film cooling since they cause larger heat transfer rates through the film to the wall than if the film were undisturbed. This paper deals with the prediction of conditions under which these long wave disturbances occur in gas-liquid flows. It will be shown that they arise because of a natural instability in the flow, and that the disturbances occurring in the gas-liquid flow system studied in this laboratory are describable by the same theory as has been used to predict the appearance of roll waves in open channel flows.

The first theoretical prediction for the conditions under which roll waves will appear was made by Jeffreys (*22*) who considered the case of turbulent flow in an open inclined channel. He used an integral momentum balance and an integral mass balance. By introducing a small disturbance into the equations for uniform flow and examining the conditions under which the disturbance will grow Jeffreys predicted for a flow obeying the Chezy resistance formula the channel inclination at which roll waves appear. Lighthill and Whitman (*28, 29*) suggested a criterion for the initiation of roll waves as the equivalence of the kinematic and dynamic wave velocities defined by them.

For the free flow of a film down a vertical wall roll waves usually occur when the flow is laminar. Unlike the case of flow in a sloping channel, gravity cannot act as a restoring force for disturbances on the liquid surface. The waves that appear are influenced by surface tension and are therefore of smaller wave length than those appearing in inclined channels. However since these flows generally involve very thin films, the wave length is much larger than the film thickness. Kapitza (*23*) used a shallow water approximation in considering their stability. He employed an integral approach in that he considered the average velocity in the film and introduced surface tension into his analysis through the pressure gradient term in the equations of motion. Yih (*39*) and Benjamin (*1*) applied the Orr-Sommerfeld equation to flow down a vertical wall. They considered solutions for small values of the product of the wave number and the Reynolds number. Since the wave number is inversely proportional to the wave length, their restriction is similar to a shallow water assumption. Benjamin obtained quite different results from Yih; however Benjamin's analysis is consistent with the results obtained by Kapitza and the theory used in this paper.

The initiation of roll waves by an air stream blowing over a horizontal flowing water film has been described by Hanratty and Engen (*15*). These experiments were conducted in a 1-in. x 12-in. channel. Water was injected on the bottom of the channel and air was blown over the water surface, the traction of the air supplying the motive force for the liquid film. The channel was long enough that the flow downstream in the water and the air were fully developed and that there was no effect of fetch on the observations. The liquid flow was controlled at a fixed rate, and the changes in the interfacial structure were noted with increasing gas flow rate. At low gas flows there were no noticeable disturbances on the liquid surface. The first disturbances to appear were a train of very small amplitude waves which extended over the whole width of the channel. With further increase in gas flow these two-dimensional waves broke up into a three-dimension pebbled structure. At high enough gas rates roll waves which extended over most of the width of the channel appeared over the pebbled surface. At first only an occasional roll wave appeared on the surface. As the gas flow increased, their frequency of appearance increased and the distance between successive roll waves decreased. At high enough gas flow rates a large number of these waves were visible on the liquid surface at any given time. Kinney, Abramson, and Sloop (*25*) and Knuth (*26*) reported the initiation of roll waves by air on a liquid film flowing along a horizontal pipe. Their experiments were conducted under flow conditions comparable to those existing in film cooled rocket motors and therefore involved much higher gas Reynolds numbers and much thinner liquid films (0.0005 to 0.005 in.) than those in the experiments reported by Hanratty and Engen.

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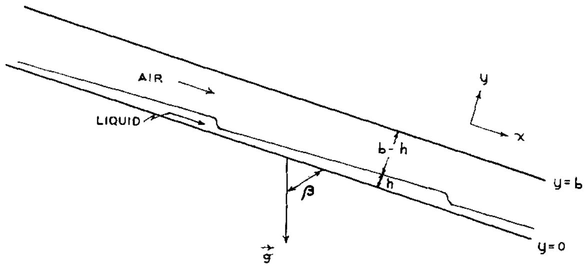

Fig. 1. Flow system treated in theory.

In this paper Jeffrey's theory will be extended to explain the initiation of roll waves by a turbulent air stream. Integral mass and momentum balances will be written for the combined air and liquid flows. Linearized perturbation theory will be used. An infinitesimal disturbance will be applied to the integral equations, and the conditions under which this disturbance will grow will be calculated by determining the neutral stability condition. As is the case in all linearized stability calculations the properties of the waves at neutral stability condition may have little relation to the properties of the disturbed flow which evolves from the initial infinitesimal disturbance. In the present calculations it is assumed that an infinitesimal sinusoidal wave grows into the highly unsymmetrical roll waves. Therefore it should not be surprising if the velocity or wave length of initial disturbance does not agree exactly with those of the observed roll waves.

The stability criterion will be developed in a general way. Then specific cases will be considered so that solutions with and without an air flow can be compared. The approximate theory will be shown to give satisfactory agreement with the exact solution of Benjamin for free flow of laminar films. The theory will be compared with the results of experiments performed in the same equipment used by Hanratty and Engen. In the previous paper on this work from this laboratory (*15*) results were presented which indicated that the roll-wave transition was not reproducible. These results have been traced to a faulty flowmeter. All of the transitions reported in this paper were easily discerned and were reproducible. In addition liquid film heights have been measured and the results have been extended to include fluids with different viscosities and surface tensions. Runs were also performed with a surface active agent which had the effect of eliminating all waves from the liquid surface. The roll-wave transition occurred on a smooth liquid film in a laminar flow. Therefore the theoretical analysis presented includes transition for the case of a smooth surface as well as a flow disturbed by waves.

The same results which are obtained by an extension of Jeffrey's theory can also be obtained by an extension of the theory of Lighthill and Whitman (*28, 29*). This equivalence of the two theories is discussed in a report by the authors (*16*).

THEORY

Development of the Integral Equations for the Liquid

Consider the flow system sketched in Figure 1. Since films having very large breadths compared with their heights will be considered, a two-dimensional incompressible flow will be assumed. An integral mass balance may be written as

$$\frac{\partial}{\partial x}\int_0^h u\,dy + \frac{\partial h}{\partial \theta} = 0 \qquad (1)$$

When one defines an average velocity as

$$u_a = \frac{1}{h}\int_0^h u\,dy \qquad (2)$$

the mass balance becomes

$$h\frac{\partial u_a}{\partial x} + u_a\frac{\partial h}{\partial x} + \frac{\partial h}{\partial \theta} = 0 \qquad (3)$$

If the wave length of the disturbances on the film are large compared with their height the flow in the y direction will be small and will have a negligible effect on the variation of the pressure. Using this shallow water assumption (*36*) one may write the following equation

$$p = p_g + (h - y)g\rho\sin\beta - \mathcal{G}\frac{\partial^2 h}{\partial x^2} \qquad (4)$$

A term involving the surface tension is included, since thin films will be considered in which the waves are of small enough wave length that surface tension will be important and yet large compared with the film thickness. For flows in which the shallow water assumption is valid it might also be assumed that the only surface stress on a liquid area element perpendicular to the direction of flow is the pressure. Using this assumption and Equation (4) one may write an integral momentum balance as follows:

$$\frac{\partial}{\partial \theta}\int_0^h u\,dy + \frac{\partial}{\partial x}\int_0^h u^2 dy = \frac{\tau_s}{\rho} - \frac{\tau_w}{\rho} + g\,h\cos\beta - \frac{h}{\rho}\frac{\partial p_g}{\partial x} - g\,h\sin\beta\frac{\partial h}{\partial x} + \frac{\mathcal{G}}{\rho}h\frac{\partial^3 h}{\partial x^3} \qquad (5)$$

When one defines a velocity profile parameter Γ as

$$\Gamma = \frac{1}{h\,u_a^2}\int_0^h u^2\,dy \qquad (6)$$

and makes use of Equation (3), the momentum balance becomes

$$\frac{\partial u_a}{\partial \theta} + (2\Gamma - 1)u_a\frac{\partial u_a}{\partial x} + (\Gamma - 1)\frac{u_a^2}{h}\frac{\partial h}{\partial x} + u_a^2\frac{\partial \Gamma}{\partial x} = \frac{\tau_s}{\rho h} - \frac{\tau_w}{\rho h} + g\cos\beta - \frac{1}{\rho}\frac{\partial p_g}{\partial x} - g\sin\beta\frac{\partial h}{\partial x} + \frac{\mathcal{G}}{\rho}\frac{\partial^3 h}{\partial x^3} \qquad (7)$$

Equation for Small Disturbances

In order to determine under what conditions roll waves will appear on the surface the behavior of a very small disturbance in a uniform film flow will be examined. An equation for such a disturbance may be obtained by substituting the following into Equations (3) and (7):

$$\begin{aligned} u_a &= \overline{u}_a + u_a' \\ h &= \overline{h} + h' \\ p_g &= \overline{p}_g + p_g' \\ \tau_w &= \overline{\tau}_w + \tau_w' \\ \tau_s &= \overline{\tau}_s + \tau_s' \end{aligned} \qquad (8)$$

The barred quantities are time averages, and the primed quantities are the disturbances on the uniform flow. The disturbances will be assumed to be small enough that second-order terms in the primed quantities may be neglected. Assuming a fully developed flow in which the time averaged quantities are not varying in the x direction and making use of the dimensionless quantities defined in the nomenclature one obtains

$$\frac{\partial \omega}{\partial X} + \frac{\partial \eta}{\partial X} + \frac{\partial \eta}{\partial \Theta} = 0 \qquad (9)$$

$$\frac{\overline{\tau}_w}{\rho\overline{h}} - \frac{\overline{\tau}_s}{\rho\overline{h}} = g\cos\beta - \frac{1}{\rho}\frac{\partial \overline{p}_g}{\partial x} \qquad (10)$$

$$\frac{\partial \omega}{\partial \Theta} + (2\Gamma - 1)\frac{\partial \omega}{\partial X} + (\Gamma - 1)\frac{\partial \eta}{\partial X}$$

$$+ \frac{\partial \Gamma}{\partial X} = -\frac{\tau_w'}{\rho \overline{u}_a^2} - \frac{\sin\beta}{N^2_{Fr}}\frac{\partial \eta}{\partial X}$$

$$- \frac{\overline{h}}{\rho \overline{u}_a^2}\frac{\partial p_g'}{\partial x} + N_{We}\frac{\partial^3 \eta}{\partial X^3} + \frac{\tau_s'}{\rho \overline{u}_a^2}$$

$$+ \eta\frac{\cos\beta}{N^2_{Fr}} - \eta\frac{\overline{h}}{\rho \overline{u}_a^2}\frac{\partial \overline{p}_g}{\partial x} \quad (11)$$

A disturbance described by the real parts of the following two equations will be studied:

$$\eta = \delta\, e^{i\,\alpha(X - C\,\Theta)} \quad (12)$$

$$\omega = \Delta\, e^{i\,\alpha(X - C\,\Theta)} \quad (13)$$

The quantity α is a real number, and the quantities δ and Δ are small enough that second-order terms in them may be neglected. The quantity C is an imaginary number:

$$C = C_r + i\,C_i \quad (14)$$

The meaning of α and C can be seen by considering the real part of Equation (12):

$$\eta_r = \delta\, e^{\alpha\, C_i\, \Theta}\cos[\alpha(X - C_r\,\Theta)] \quad (15)$$

This describes a sinusoidal wave of amplitude $\delta\, e^{\alpha\, C_i\, \Theta}$, of velocity C_r, and of wave length:

$$\lambda = \frac{2\,\pi\,\overline{h}}{\alpha} \quad (16)$$

If C_i is greater than zero, the wave amplitude will increase with time; if it is less than zero, the wave amplitude will decrease with time. The condition of $C_i = 0$ known as *neutral stability* will be used as the criterion for the appearance of roll waves on a uniform surface.

If Equations (12) and (13) are substituted into Equation (9), the following relation is obtained between Δ and δ:

$$\Delta = \delta\,(C - 1) \quad (17)$$

Equations (12) and (13) will be substituted into Equation (11) making use of Equation (17) and the resulting expression solved for the case $C_i = 0$. However before this can be done expressions for τ_w', p_g', and τ_s' must be obtained in terms of η and ω.

Evaluation of $\overline{\tau}_w$ and τ'_w

For disturbances of very long wave length the change of height with distance is gradual enough that the shape of the velocity profile would be closely approximated by that for a uniform flow. It will therefore be assumed that wall shear stress τ_w is related to u_a and h in the same manner as for a flow without roll waves present. Depending on the situation the uniform flow prior to the appearance of roll waves may be either a steady laminar flow or an unsteady flow in which there exists on the surface of the water a pattern of waves of much smaller length than the roll waves.

For a uniform laminar flow the Navier-Stokes equations may be solved to yield

$$\tau_w = \frac{2\,\mu\,u_a}{h} - \left(\frac{dp}{dx} - \rho g \cos\beta\right)\frac{h}{3} \quad (18)$$

For the case of free flow, one in which there is no gas flow

$$\frac{dp}{dx} = 0 \quad (19)$$

$$\tau_w = h\,\rho g \cos\beta \quad (20)$$

Substituting Equations (19) and (20) into Equation (18) one obtains

$$\tau_w = \frac{3\,\mu\,u_a}{h} \quad (21)$$

Substituting Equation (8) into Equation (21) and neglecting second-order terms in the disturbances one gets

$$\frac{\tau_w'}{\rho\,\overline{u}_a^2} = \frac{3}{N_{Re}}\,\omega - \frac{3}{N_{Re}}\,\eta \quad (22)$$

Since

$$\frac{3\,\mu\,u_a}{h} = h\,\rho g \cos\beta \quad (23)$$

Equation (22) may also be written in terms of the Froude number by using the expression

$$\frac{N_{Re}}{3} = \frac{N^2_{Fr}}{\cos\beta} \quad (24)$$

For the case in which air is flowing over the liquid surface the pressure-gradient term may be eliminated from Equation (18) through the following force balances in the liquid and the gas:

$$h\frac{\partial p}{\partial x} = \tau_s + \rho g\,h\cos\beta - \tau_w \quad (25)$$

$$\tau_s(1 + \gamma) = -\frac{\partial p}{\partial x}(b - h) \quad (26)$$

The term γ in the above expression arises since the shear stress at the surface of the liquid, τ_s, might not equal the shear stress at the surface without liquid, τ_o, and is defined by the equation

$$\tau_o = \gamma\,\tau_s \quad (27)$$

Eliminating τ_s between Equations (25) and (26) and substituting the resulting expression for the pressure gradient into Equation (18) one gets an expression for τ_w in terms of u_a and h:

$$\tau_w = \frac{3\,\mu\,u_a}{h}\left[1 - \frac{(b-h)}{2\,h(1+\gamma) + 3(b-h)}\right] + \rho g\,h\cos\beta\left[\frac{(b-h)}{2h\,(1+\gamma) + 3(b-h)}\right] \quad (28)$$

Using Equation (8) one may evaluate the fluctuating wall shear stress:

$$\frac{\tau_w'}{\rho\,\overline{u}_a^2} = \omega\frac{3}{N_{Re}}\left\{1 - \frac{1}{2(1+\gamma)\,H + 3}\right. + \eta\frac{3}{N_{Re}}\left\{-1 + \frac{H+1}{2(1+\gamma)\,H+3} + \frac{(2\gamma - 1)\,H}{[2(1+\gamma)\,H + 3]^2}\right\} + \eta\frac{\cos\boldsymbol{\beta}}{N^2_{Fr}}\left\{\frac{1-H}{2(1+\gamma)\,H+3} - \frac{(2\gamma-1)\,H}{[2(1+\gamma)\,H+3]^2}\right\} \quad (29)$$

For free flow down an inclined plane the flow is usually laminar under conditions for which roll-wave transition occurs. However for small inclinations of the plane, that is for large β, the flow is turbulent at roll-wave transition. For such flows the wall shear stress will be taken as

$$\tau_w = A\,2\,\mu\frac{u_a}{h} \quad (30)$$

where the factor A is a function of the liquid Reynolds number and the condition of the surface of the plane. For a smooth surface obeying the Blasius resistance law

$$A = 0.0396\,N_{Re}^{0.75} \quad (31)$$

For a completely rough solid surface

$$A = \text{constant} \times N_{Re} \quad (32)$$

A relation for the fluctuation in the shear stress can be obtained from Equations (30) and (8):

$$\frac{\tau_w'}{\rho\,\overline{u}_a^2} = \frac{2\,A}{N_{Re}}(\omega - \eta) + 2\frac{d\,A}{dN_{Re}}(\eta + \omega) \quad (33)$$

For situations involving the flow of air and a liquid, unsteady flows with a wavy structure can exist at much lower Reynolds numbers than one generally associates with turbulence based on knowledge of the free flow of liquid films. Colburn (7) in considering the effect of vapor velocity on the performance of condensers suggested that equations describing the velocity distribution for turbulent flows near a wall might approximate the velocity distribution in film flows. This suggestion has been developed by Kinney, Abramson, and Sloop (*25*) and by Dukler (*12*). Experiments performed by Hershman (*18*) in a horizontal channel, $\beta = 90$ deg., have demonstrated that the hypothesis is valid in the region where roll-wave transition occurs for such a system. For the concurrent flow in a horizontal channel with a smooth bottom the Colburn hypothesis suggests that the coefficient A is a unique function of Reynolds number as in the free flow case. However for gas-liquid flows on an inclined or a vertical wall the coefficient A would probably be a function of Froude number also as is suggested by Equations (28) and (29) and by the calculations of Dukler (*12*). However since experimental verification of the hypothesis is available only for the horizontal flow, this will be the only case treated in this paper.

Velocity data for turbulent flow have been correlated by an equation of the form

$$\frac{u}{u^*} = f(y^+) \tag{34}$$

If this equation is assumed to describe the flow in a liquid film

$$u_a = \frac{u^*}{h} \int_0^{h^+} \frac{\mu}{\rho u^*} f(y^+)\, dy^+ \tag{35}$$

$$N_{Re} = \int_0^{h^+} f(y^+)\, dy^+ \tag{36}$$

Therefore h^+ is a unique function of the Reynolds number. From the definition of h^+

$$\frac{\tau_w}{\rho} = \frac{h^{+2}\,\mu^2}{h^2\,\rho^2}$$

$$= \frac{h^{+2}}{2\,N_{Re}}\,\frac{\mu}{\rho}\,\frac{2\,u_a}{h} \tag{37}$$

Comparing Equations (37) and (30) one gets

$$A = \frac{h^{+2}}{2\,N_{Re}} \tag{38}$$

The fluctuations in the wall shear stress can be calculated from Equations (33) and (38). For the calculations presented in this paper the velocity profile equations presented by Deissler (*10*) were used to evaluate the relation between h^+ and N_{Re}. The details of this calculation are presented by Hershman (*18*).

Evaluation of Γ and dΓ/dX

The velocity profile parameter Γ can be evaluated from Equation (6) with the velocity profile obtained from a solution to the Navier Stokes equations for laminar flow or by Equation (34) for the unsteady flows treated in this paper. For laminar flow

$$\Gamma = \frac{36}{20} - \frac{6}{20}\,\frac{\tau_w\,h}{\mu\,u_a} + \frac{1}{30}\,\frac{\tau_w^{\,2}\,h^2}{\mu^2\,u_a^{\,2}} \tag{39}$$

For a free flow substitution of Equation (21) yields

$$\Gamma = \frac{6}{5} \tag{40}$$

Equation (28) and (39) could be used to obtain Γ for a gas-liquid flow. The quantity $d\Gamma/dX$ can be obtained by differentiating the expression for Γ and will be represented by an equation of the form

$$\frac{d\Gamma}{dX} = \Gamma_\omega\,\frac{d\omega}{\partial X} + \Gamma_\eta\,\frac{d\eta}{\partial X} \tag{41}$$

If the flow is to be described by Equation (34) it can be shown that

$$\Gamma = \frac{h^+}{N^2_{Re}} \int_0^{h^+} f^2(y^+)\, dy^+ \tag{42}$$

Since h^+ is a unique function of N_{Re}, then Γ is also a unique function of N_{Re}. Therefore

$$\Gamma_\omega = \Gamma_\eta = \frac{d\Gamma}{d\,N_{Re}}\,N_{Re} \tag{43}$$

Evaluation of ∂p'g/∂X

The pressure changes in a gas flow caused by a wavy boundary have been calculated by Kelvin and Helmholtz (*27*), by Jeffreys (*21*), and more recently by Miles (*31, 32*) and Benjamin (*2*). These theories treat a gas of infinite extent flowing over a wavy liquid surface. They may be discussed by means of an equation of the following form:

$$p_g' = \overline{u_a}^2 (d + i\,l)\,\delta\,\alpha\,\rho_g \left(\frac{1}{V} - C_r\right)^2 l^{i\,\alpha(X - C_r\,\Theta)} \tag{44}$$

The real part of the above would be

$$p'_{gr} = \overline{u_a}^2\ \delta\,\alpha\,\rho\,\rho_g \left(\frac{1}{V} - C_r\right)^2 [d \cos \alpha(X - C_r\,\Theta) - e \sin \alpha(X - C_r\,\Theta)] \tag{45}$$

A comparison of this expression with Equation (15) for the case of $C_i = 0$ shows that the factor d is associated with the component of the pressure which has a minimum or maximum when η is a maximum and that the factor e is associated with the component of pressure which has a maximum or minimum when the wave slope is a maximum. By solving the equations of motion for the case of zero viscosity for a uniform velocity field Kelvin-Helmholtz obtained $d = -1$ and $e = 0$. By considering observations of the wind velocity at which waves are generated on a water surface Jeffreys suggested a value for e of 0.3. Miles and Benjamin have presented more rigorous derivations for these coefficients. The air-water results presented in this paper represent a situation different from that treated in the above papers in that the gas phase above the liquid is enclosed and is not of infinite extent. In fact the wave lengths of roll waves which first appeared were large compared with the thickness of the gas space above them. If the Kelvin-Helmholtz analysis were applied to the system considered in this paper, it would yield

$$e = 0$$

$$d = -\frac{1}{\tanh\left(\dfrac{\alpha}{H}\right)} \tag{46}$$

For small values of α/H the above reduces to

$$d = -\frac{H}{\alpha} \tag{47}$$

An expression similar to that suggested by Kelvin-Helmholtz and by Jeffreys can be obtained in a somewhat different way for the experiments considered in this paper. Since for these experiments the wave length of the initial disturbance was quite large compared with the thickness of the air space, a shallow gas assumption can be made similar to that for the liquid flow. Proceeding with such an analysis the following two equations can be derived from the momentum balance and the mass balance:

$$\frac{\overline{h}}{\rho\,\overline{u_a}^2}\,\frac{\partial p_g'}{\partial x} = -R\,\frac{\partial \omega_g}{\partial \Theta} - \frac{R(2\,\Gamma_g - 1)}{V}\,\frac{\partial \omega_g}{\partial X} - (1 - \Gamma_g)\,\frac{R\,H}{V^2}\,\frac{\partial \eta}{\partial X} - \frac{R}{V^2}\,\frac{\partial \Gamma_g}{\partial X} - \frac{H\,\tau_s'\,(\gamma + 1)}{\rho\,\overline{u_a}^2} - \frac{(\gamma + 1)\,H^2\,\overline{\tau_s}}{\rho\,\overline{u_a}^2}\,\eta \tag{48}$$

$$\frac{V}{H}\,\frac{\partial \omega_g}{\partial X} - \frac{\partial \eta}{\partial X} - \frac{\partial \eta}{\partial \Theta}\,V = 0 \tag{49}$$

The dimensionless disturbance velocity in the gas phase will be assumed to be

$$\omega_g = \Delta_g\,e^{i\,\alpha(X - C\,\Theta)} \tag{50}$$

Substituting Equations (12), (13), and (50) into Equation (49) one gets

$$\Delta_g = \delta\left(\frac{H}{V} - C\,H\right) \quad (51)$$

From Equations (48), (12), (13), (17), (50), (51), and (58)

$$\frac{h}{\rho\,\overline{u}_a^2}\,\frac{\partial p_g'}{\partial x} = -\,i\,\alpha\,\frac{\eta\,R\,H}{V^2}\,[\Gamma_g - 2\,V\,C + V^2\,C^2] - \frac{B\,N_{Re}{}^n{}_g\,(\gamma + 1)\,\eta\,R\,H^2}{V^2}\,[2 - (2 + n)\,C\,V + H] \quad (52)$$

Comparing Equations (44) and (52) one obtains

$$\frac{d}{V^2}\,(1 - 2\,V\,C + V^2\,C^2) = -\,\frac{H}{\alpha\,V^2}\,[\Gamma_g - 2V\,C + V^2\,C^2] \quad (53)$$

$$\frac{e}{V^2}\,(1 - 2\,V\,C + V^2\,C^2) = \frac{(\gamma + 1)\,B\,N_{Re}{}^n{}_g\,H^2}{\alpha^2\,V^2}\,[2 - (2 + n)\,C\,V + H] \quad (54)$$

The shape factor for the gas velocity profile Γ_g has a value of about 1.02. For $\Gamma_g = 1$, Equation (53) is identical to Equation (47). For $C\,V$ and H small compared with 2 or Γ_g

$$d = -\,\frac{\Gamma_g\,H}{\alpha} \quad (55)$$

$$e = \frac{2(\gamma + 1)\,B\,N_{Re}{}^n{}_g\,H^2}{\alpha^2} \quad (56)$$

Evaluation of τ_s and τ'_s

If the wave length of the roll waves is large compared with the thickness of the gas space, the same sort of argument can be used to evaluate τ_s and τ_s' as were used to evaluate τ_w and τ_w'. Since the gas flow is turbulent, an equation of the following form is used to represent the surface stress:

$$\frac{\tau_s}{\rho_g\,u^2_{ga}} = B\,N_{Re}{}^n{}_g \quad (57)$$

If Equations (8) are substituted into Equation (57) and if second-order terms are neglected

$$\frac{\tau_s'}{\rho\,\overline{u}_a^2} = \frac{(2 + n)\,B\,R}{V}\,N_{Re}{}^n{}_g\;\omega_g - \frac{n\,B\,R\,H}{V^2}\,N_{Re}{}^n{}_g\;\eta \quad (58)$$

ROLL-WAVE TRANSITION FOR FREE FLOWS

The equations which have been developed become greatly simplified for free flow down an inclined plane, since

$$\frac{\partial p_g}{\partial x},\ \frac{\partial p_g'}{\partial x},\ \tau_s',\ \frac{d\Gamma}{dx} = 0 \quad (59)$$

If Equations (12), (13), (22), (40), and (59) are substituted into Equation (11), the following expression is obtained for a situation with laminar flow:

$$C^2 + C\left[-\frac{12}{5} + \frac{i}{\alpha}\,\frac{\cos\beta}{N^2_{Fr}}\right] + \left[\frac{6}{5} - \frac{3i}{\alpha}\,\frac{\cos\beta}{N^2_{Fr}} - \frac{\sin\beta}{N^2_{Fr}} - \alpha^2\,N_{We}\right] = 0 \quad (60)$$

By equating the real and the imaginary parts of the above expression to zero two equations are obtained. For the case of $C_i = 0$

$$C_r = 3 \quad (61)$$

$$C_r^2 - \frac{12}{5}\,C_r + \frac{6}{5} - \frac{\sin\beta}{N^2_{Fr}} - \alpha^2\,N_{We} = 0 \quad (62)$$

Substituting for C_r one obtains the following expression for neutral stability condition:

$$\frac{\sin\beta}{N^2_{Fr}} = 3 - \alpha^2\,N_{We} \quad (64)$$

These results compare favorably with the results obtained by Benjamin (*1*) from a solution of the Orr-Sommerfeld equation for small values of $\alpha\,N_{Re}$, thereby giving support to the approximate analysis with integral equations. Restating his criteria for transition in the nomenclature used in this paper one gets

$$C_r = 3$$

$$\frac{\sin\beta}{N^2_{Fr}} = \frac{18}{5} - \alpha^2\,N_{We} \quad (65)$$

For flow down a vertical wall Equations (61) and (62) are applicable with

$$C_r = 3$$

$$N_{We} = \frac{3}{\alpha^2} \quad (66)$$

Making use of Equations (24) and (16) one can write the above criterion in an alternate form:

$$N_{Re}{}^{5/3} = \left(\frac{\overline{h}}{\lambda}\right)^2 \left[\frac{\pi^2\,\frac{4}{3}\,\sigma\,\rho^{1/3}\,3^{1/3}}{\mu^{4/3}\,g^{1/3}}\right] \quad (67)$$

For water of 1 centipoise the above equation becomes

$$N_{Re}{}^{5/3} = 63.9 \times 10^3 \left(\frac{\overline{h}}{\lambda}\right)^2 \quad (68)$$

This expression is plotted in Figure 2. It will be noted that although the values of $\lambda/\overline{h}$ are large, the wave length λ is small enough for surface tension to be having an effect. The region above the curve is the region in which disturbances will be amplified. Equation (68) predicts for a given Reynolds number the minimum wave length for unstable disturbances. Binnie (*3*) has studied the initiation of waves on a water film flowing down the outside of a long vertical tube. He found that transition occurred at $N_{Re} = 4.4$ with waves having $C_r = 3.5$ and $\lambda/\overline{h} = 102$. If $N_{Re} = 4.4$ is substituted into Equation (68), a value of $\lambda/\overline{h} = 74$ is calculated. The observed wave velocity and wave length are in approximate agreement with the theory presented; however the theory does not predict any definite transition condition. Equations (67) and (68) do not predict a minimum value of N_{Re}; they merely limit the range of values of wave length of unstable disturbances. Numerous other investigators (*20*) have examined this transition for a wide variety of liquids and reported critical values of N_{Re} ranging from 1.6 to 6.2. Benjamin (*1*) has explained the observed critical Reynolds num-

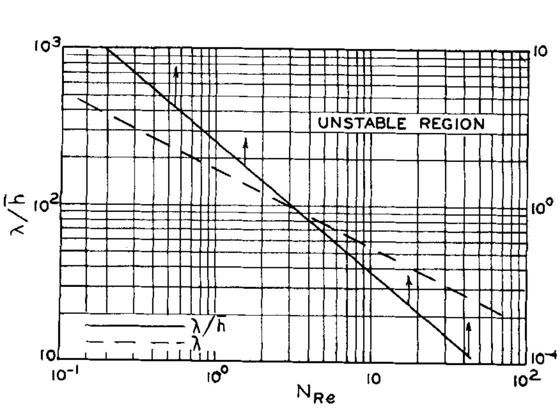

Fig. 2. Neutral stability curve for film flow down a vertical wall.

bers by the fact that at small Reynolds numbers the growth rate of unstable disturbances is so small that they did not grow to a large enough size to be observed in the experimental setups used. Kapitza (23) has carried out a second-order solution to the problem in that he kept terms of order δ^2. He calculated the surface energy of the predicted wave train and found that the surface energy decreased with increasing wave length, went through a minimum, and then increased with increasing wave length. Kapitza interpreted the increase of surface energy at large wave lengths as resulting from a wrinkling of the crests of the waves. He reasoned that such waves would be unstable and suggested that the wave length at which this minimum in the surface energy occurred would be an upper limit for the disturbance wave length. On this basis

$$\left(\frac{\lambda}{h}\right)_{\substack{\text{upper}\\ \text{limit}}} = 13.3\ N_{Re} \quad (69)$$

If this is substituted into Equation (67), the critical condition becomes

$$N_{Re}^{11/3} = 0.108\ \frac{\sigma\ \rho^{1/3}}{\mu^{4/3} g^{1/3}} \quad (70)$$

For water of 1-centipoise transition is predicted at $N_{Re} = 5.0$.* This is in good agreement with experiment. However owing to the approximate nature of the solution upon which Kapitza's observations are based his theory is not conclusive despite the agreement with experiment. The question as to whether a critical Reynolds number exists for roll-wave transition on free vertical films has not been answered conclusively. Within the limits of linearized stability theory Benjamin's conclusions are valid if he has accounted for all the physical variables of the problem.† Whether finite disturbances introduced by the liquid entry is responsible for the observed critical Reynold's numbers is not yet known.

ROLL-WAVE TRANSITION CONCURRENT GAS-LIQUID FLOW

Equation (11) with $\sin\beta = 1$ and $\cos\beta = 0$ may be used to predict the conditions for roll-wave transition for concurrent gas-liquid flow in a horizontal plane. Equations (12) and (13) are substituted for η and ω, Equation (44) for p_g', Equation (58) for τ_s'. The average pressure gradient $\partial \overline{p}_g/\partial x$ is related to the average surface shear stress $\overline{\tau_s}$ by Equations (26) and (27). For roll-wave transition over an unsteady surface Equations (33), (38), (41), and (42) are used to calculate τ_w', Γ, and $d\Gamma/dX$. Making these substitutions one obtains the following equations for the case of $C_i = 0$:

$$C_r = \frac{2}{1 + \dfrac{N_{Re}}{A}\dfrac{dA}{dN_{Re}}} + \frac{N_{Re}\,\alpha^2 e R\,(1 - C_r V)^2}{2AV^2\left(1 + \dfrac{N_{Re}}{A}\dfrac{dA}{dN_{Re}}\right)}$$

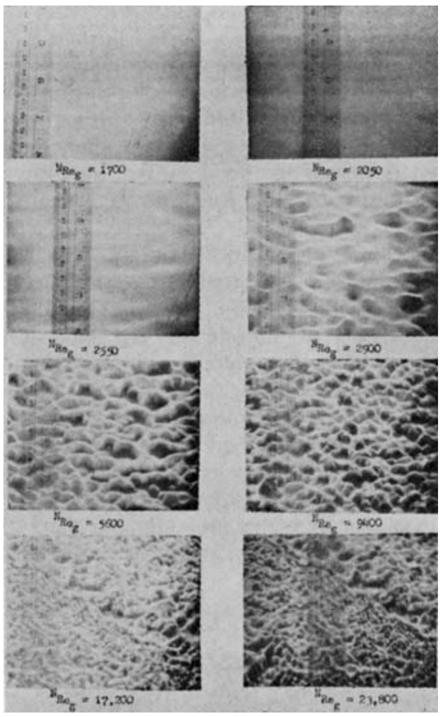

Fig. 3. Waves on water surface as N_{Re_g} increased.

$$+ \frac{\overline{\tau_s}\,H(3 + \gamma - n\,C_r V)}{\overline{\tau_w}\left(1 + \dfrac{N_{Re}}{A}\dfrac{dA}{dN_{Re}}\right)} \quad (71)$$

$$\frac{\overline{\tau_w}}{\rho\,\overline{u_a}^2} = \frac{2A}{N_{Re}} \quad (72)$$

$$\frac{\overline{\tau_w}}{\overline{\tau_s}} = 1 + (\gamma + 1)\,H \quad (73)$$

$$C_r^2 + C_r(-2\Gamma - \Gamma_w) + \Gamma - \frac{1}{N^2_{Fr}} - \frac{\alpha R d}{V^2}[1 - C_r V]^2 - \alpha^2 N_{We} = 0 \quad (74)$$

It is to be noted that only the pressure component in phase with the wave slopes affects the wave velocity. On the other hand this pressure component affects the stability condition, Equation (74), only because of its effect on velocity. Since this effect is relatively small, the above analysis indicates that only the pressure component in phase with the wave amplitude is important in determining roll-wave transition. For small H and V Equation (71) and (73) become

$$C_r = \frac{2 + H\,(\gamma + 3)}{1 + \dfrac{N_{Re}}{A}\dfrac{dA}{dN_{Re}}} \quad (75)$$

$$C_r^2 + (-2\Gamma - \Gamma_w) + \Gamma - \frac{1}{N^2_{Fr}} - \frac{\alpha R d}{V^2} - \alpha^2 N_{We} = 0 \quad (76)$$

It is of interest to examine these two equations in the limit as $H \rightarrow 0$. This would be obtained as $N_{Re} \rightarrow 0$. For this limit

$$C_r = 2 \quad (77)$$

$$\frac{1}{N^2_{Fr}} + \frac{\alpha R d}{V^2} + \alpha^2\ N_{We} = 0 \quad (78)$$

The pressure fluctuation in phase with the amplitude arises because of the changes of velocity in the gas phase. When the amplitude is a maximum, the gas velocity has increased and the pressure has decreased. Therefore it may be viewed as a suction on the wave crests. Equation (78) expresses a balance between this suction and the restoring force due to gravity and surface tension. If Equation (46) is used for d, Equation (78) may be written as follows:

$$\overline{u}_{ga} = \left[\left(\frac{1}{R}\right)\left(\frac{\lambda g}{2\pi} + \frac{2\pi\sigma}{\lambda\rho}\right)\left(\tanh\frac{2\pi b}{\lambda}\right)\right]^{1/2} \quad (79)$$

The minimum $\overline{u}_{ga}$ according to Equation (79) is when $\lambda = 1.70$ cm. for an air-water system with a b of 1 in. The critical $\overline{u}_{ga}$ is 21.9 ft./sec. However the wave length of the roll waves observed at transition are much larger than 1.70 cm. Therefore it is not to be expected that Equation (79) represents a proper limit for the experimental measurements reported in this paper. As has been mentioned earlier in the paper a shallow gas assumption appears to be in closer agreement with observations. If the coefficient d is defined by Equation (55) and if the effect surface tension is neglected, the transition gas velocity is given as

$$\overline{u}_{ga} = \left[\frac{\Gamma_g\,b\,g}{R}\right]^{1/2} \quad (80)$$

* Kapitza obtained N_{Re} = 5.8 at transition from his stability criterion.

† Sternling and Barr-David discuss this point in a recent manuscript submitted to the *A.I.Ch.E. Journal*.

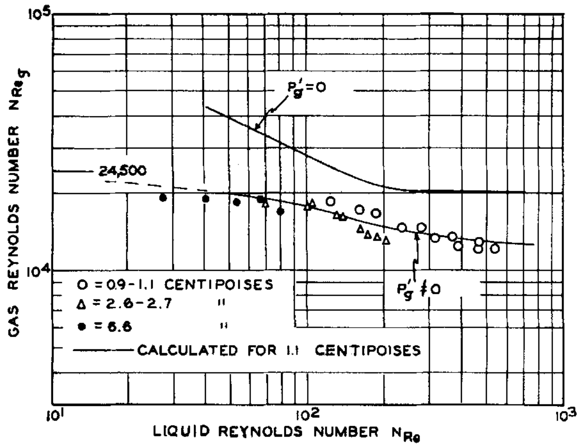

Fig. 4. Roll-wave transition for water and glycerine-water solutions.

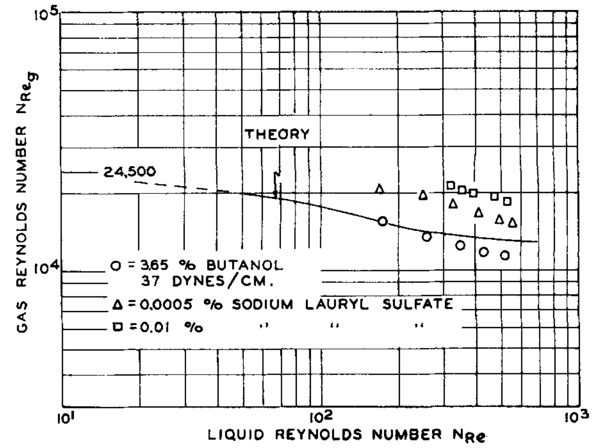

Fig. 5. Effect of surface tension and surface active agent on roll-wave transition.

For the conditions of the experiments reported in this paper this would give $\bar{u}_{ga} = 50$ ft./sec. and $N_{Re_g} = 24{,}500$.

If Equations (28), (29), and (39) are used for τ_w, τ_w', and Γ with $\cos\beta = 0$ and $\sin\beta = 1$, then equations analogous to (71), (72), (74) can be derived for roll-wave transition over a smooth horizontal surface. The limiting solution for $H \to 0$ is the same as for roll-wave transition over an unsteady surface.

MEASUREMENTS OF GAS-LIQUID ROLL-WAVE TRANSITION

Experiments were carried out to study the roll-wave transition for a horizontal air-liquid flow with liquids whose viscosities varied from 1 to 6.60 centipoises and whose surface tensions varied from 35 to 75 dynes/cm. This was accomplished by using glycerine-water solutions and butanol-water solutions. Measurements were also obtained with 0.0005 to 0.01% solutions of sodium lauryl sulfate. The details of the experiments are described by Hershman (*18*). They were conducted in equipment designed by Hanratty and Engen (*15*). Liquid and air flowed concurrently in a 20 ft. long by 1 ft. wide by 1 in. high horizontal rectangular channel made of Plexiglas. Roll-wave transition was determined at fixed liquid rates by varying the gas rate. Transition was noted by the appearance of waves of the type shown in Figure 7 of reference *15*. A series of overhead flash photographs showing the change of surface structure for an air-water flow are shown in Figure 3. The roll waves appeared to form very close to the entry, and they covered very nearly the whole breadth of the channel. Changes in the design of the liquid entry did not affect the transition conditions (*30*). No significant difference was noted in the surface structure between the runs with water and the runs with butanol solutions. However in the runs with solutions of sodium lauryl sulfate the surface was completely smooth and the flow was determined to be laminar by tracer studies (*18*) at the roll-wave transition.

The average liquid height was determined by a technique similar to that used by Von Rossum (*38*). It consisted of measuring the electrical resistance of a length of the liquid film (*17*). The resistance measurements were compared with a calibration curve obtained from systems on which there were no surface disturbances. Although experiments conducted in a pan of water indicated no marked difference in the readings with and without a disturbed surface, the film heights reported in this paper are possibly open to errors of the magnitude of the surface waves. The value of γ was calculated from the distortion of the gas velocity profile (*15, 18*). The coefficient γ had a value of unity when the liquid surface was smooth and was as low as 0.4 for some of the disturbed surfaces. For some runs γ was determined from measurements of the velocity profile in the gas flow; however for most of them it was calculated by comparing pressure drops measured with liquid flowing to those measured in a dry channel. The details of this calculation are presented by Hershman (*18*). The shear stress at the liquid surface τ_s was calculated with

TABLE 1. ROLL-WAVE TRANSITION ON A ROUGH SURFACE

N_{Re}	N_{Re_g}	u_g^* (ft./sec.)	$\frac{HR}{V^2}$	C_r (calc.)	N_{Fr} (calc.)	N_{Fr} (exp.)
Liquid—water						
ρ = 62.4 lb./cu. ft.			μ = 1.08 centipoise			
124	18,700	2.45	0.093	1.669	2.61	2.73
316	13,200	2.29	0.102	1.606	2.10	1.97
468	12,100	2.27	0.097	1.554	2.08	1.93
Liquid—18.8% glycerine in water						
ρ = 64.6 lb./cu. ft.			μ = 1.70 centipoise			
80	19,600	2.72	0.135	1.773	2.21	2.39
205	14,000	2.36	0.123	1.714	1.83	1.87
306	12,100	2.33	0.114	1.684	1.77	1.76
Liquid—36.0% glycerine in water						
ρ = 67.4 lb./cu. ft.			μ = 2.60 centipoise			
68	18,200	2.94	0.231	1.830	1.73	1.80
138	16,000	2.76	0.219	1.811	1.52	1.62
204	13,000	2.52	0.241	1.814	1.37	1.37
Liquid—55.3% glycerine in water						
ρ = 70.3 lb./cu. ft.			μ = 6.60 centipoise			
27	19,200	3.39	0.903	2.023	0.95	1.00
53	18,400	3.52	0.605	2.012	1.06	1.20
79	17,000	3.46	0.432	2.000	1.11	1.38
Liquid—3.65% butanol in water						
ρ = 62.4 lb./cu. ft.			μ = 0.99 centipoise		σ = 37.0 dynes/cm.	
174	15,500	2.35	0.100	1.687	2.16	2.16
426	11,800	2.22	0.100	1.568	2.07	1.78
514	11,400	2.20	0.091	1.541	2.08	1.78

Table 2. Roll-Wave Transition on a Smooth Surface

N_{Re}	N_{Re_g}	u_g^* (ft./sec.)	$\frac{HR}{V^2}$	Γ (calc.)	C_r (calc.)	N_{Fr} (calc.)	N_{Fr} (exp.)
Liquid—0.0005% sodium lauryl sulfate in water							
ρ = 62.4 lb./cu. ft.			μ = 1.02 centipoise				
169	20,400	2.23	0.100	1.327	2.110	1.90	2.75
329	17,900	2.18	0.080	1.324	2.145	1.75	2.75
552	15,000	2.12	0.064	1.320	2.195	1.57	2.66
Liquid—0.010% sodium lauryl sulfate in water							
ρ = 62.4 lb./cu. ft.			μ = 1.06 centipoise				
320	21,100	2.28	0.084	1.325	2.133	1.92	3.24
397	19,900	2.30	0.074	1.324	2.148	1.75	3.29
530	18,400	2.37	0.066	1.320	2.183	1.64	3.20

Equation (28) with measurements of the pressure gradient, the film height, and the distortion factor γ.

Data from runs with water and water-glycerine solutions are presented in Figure 4. The effect of increasing the fluid viscosity is to decrease slightly the gas velocity required for transition to roll waves. Theoretical curves for transition with water are presented, for the case of $P_g' = 0$ and for P_g' defined by Equations (44) and (55). There is a marked difference between the two theoretical curves. The curve with $P_g' = 0$ goes to infinity as the liquid Reynolds becomes smaller, while the theoretical curve with $P_g' \neq 0$ goes to the asymptotic solution described by Equation (80). Comparison of the experimental data with theory not only supports the theory but also clearly indicates the important role which gas phase pressure fluctuations were playing in the experiments reported. Although not shown in Figure 4 calculations for larger viscosities than water show the same trend as indicated by the data. At lower liquid Reynolds numbers the low viscosity and high viscosity calculations come into the same asymptote. As the liquid Reynolds number is increased, the calculated curve with a high viscosity fans out below the calculations for water. Data for butanol solutions and for solutions of sodium lauryl sulfate are shown in Figure 5. The theoretical line for water is replotted from Figure 4. Comparison of the data in Figures 4 and 5 shows that a decrease in the interfacial tension did not affect the transition markedly. The addition of the surface active agent sodium lauryl sulfate not only decreases the surface tension but also increases the surface viscosity (*5, 6, 13*). It is this increase in the surface viscosity which is responsible for the damping of surface disturbances. The data presented in Figure 5 show that the addition of surface active agents inhibit slightly the roll-wave transition; however it is apparent that roll waves are not affected so much by the addition of surface active agents as are other types of waves.

The comparison of theory and experiment in Figure 4 depended on the assumption of a relation between the gas phase friction factor and liquid Reynolds number and of the velocity profile distortion coefficient γ and liquid Reynolds number. A more direct comparison is made in Tables 1 and 2, where the conditions at roll-wave transition are tabulated and calculated Froude numbers are compared with measured Froude numbers. For the runs listed in Table 1 the Froude numbers were calculated with Equations (75) and (76). Satisfactory agreement between theory and experiment was obtained. For the runs listed in Table 2 theoretical equations developed for roll-wave transition over a smooth surface were used. Transition occurred at higher Froude numbers than predicted. The exact reason for this lack of agreement is not known. Since the transition occurred over a smooth undisturbed surface, it might be that a length of channel is needed for the initial growth of small disturbances. Likewise the surface film could inhibit the initial growth of disturbances.

Calculated wave velocities are compared with measured wave velocities in Table 3. The velocity was measured by timing the travel of a wave over a given length with a stop watch and is therefore accurate to about 10%. There is approximate agreement between the calculated and the measured values; however the measured velocities appear to be somewhat higher. This could be explained by the fact that the calculations are based on waves of infinitesimal size at the neutral stability condition. Observed waves are of finite size and are at gas velocities larger than those at neutral stability.

In comparing experimental and calculated roll-wave transitions in Table 1 the Froude number at transition was calculated from experimental values of H, V, N_{Re}, γ and HR/V^2. It might be questioned that the agreement between experiment and theory resulted from the fact that calculated and experimental Froude number would be the same for almost any possible set of experimental conditions, since so many experimental measurements were used in the calculations. Table 4 has been constructed to illustrate that this is not the case. Experimental conditions during a run to determine the roll-wave transition are tabulated. The Froude number for transition calculated for each of the sets of experimental conditions is presented. Prior to transition the measured Froude number is below that predicted by theory; it becomes larger

Table 3. Roll-Wave Velocities

N_{Re}	N_{Re_g} (at transition)	N_{Re_g} (for velocity measurement)	u_a (ft./sec.)	C_R (exp.)	C_R (calc.)
Liquid—water					
ρ = 62.4 lb./cu. ft.		μ = 0.93 centipoise			
543	12,200	12,900	0.942	1.6	1.52
370	13,600	14,200	0.810	1.9	1.57
186	16,900	19,700	0.863	2.2	1.46
Liquid—3 to 5.0% butanol in water					
ρ = 62.4 lb./cu. ft.		μ = 1.03 centipoise			
354	12,800	12,800	0.785	1.6	1.54
354	12,800	13,500	0.814	1.5	1.59
Liquid—53.0 to 55.3% glycerine in water					
ρ = 70.3 lb./cu. ft.		μ = 6.60 centipoise			
53	18,400	18,500	0.542	2.3	2.03
40	18,900	18,900	0.504	2.4	2.02
27	19,200	19,200	0.378	2.9	2.04
Liquid—0.0005% sodium lauryl sulfate in water					
ρ = 62.4 lb./cu. ft.		μ = 0.90 centipoise			
562	15,400	18,600	1.46	2.1	2.18
453	15,900	18,600	1.35	2.3	2.21

than that predicted by theory after the observed transition condition.

WAVE FORMATION IN FILM COOLANTS

Visual experiments have been carried out by Kinney, Abramson, and Sloop (*25*) and by Knuth (*26*) to determine when long wave length disturbances will appear on liquid-coolant films on the inner surfaces of horizontal tubes containing a high velocity air stream. The experiments by Kinney, et al. were conducted in 4- and 2-in. tubes over a gas Reynolds number range of 4.1×10^5 to 29×10^5 with water, water-detergent solutions, and aqueous ethylene glycol solutions. Knuth's experiments were conducted in a 3-in. tube over a gas Reynolds number range of 3.07×10^5 to 5.95×10^5. These experiments showed that liquid Reynolds numbers at transition to roll waves are independent of the gas Reynolds number over the range investigated but depend on liquid viscosity in that the lower liquid Reynolds number required the higher the liquid viscosity. For water the transition occurred at a liquid Reynolds number of about 30, and for a liquid with a viscosity about 2½ times as large as water transition occurred at a liquid Reynolds number of about 10.

These experiments are at much higher gas velocities than used in the experiments reported in this paper. In fact they would be in the region associated with the limiting solution given by Equations (77) and (78). However the calculated results presented in this paper should not be applicable to the film coolant studies owing to marked differences in the geometry of the two systems. For the runs inside a horizontal tube there could exist a variation of the thickness of the liquid film around the circumference of the tube. Even more of a difference is the fact that the component of gravity normal to the surface of the liquid varies from $+g$ at the bottom to zero at the sides to $-g$ at the top of the tube. On the bottom of the tube gravity acts as a stabilizing influence, whereas on the top it acts as a destabilizing influence. Whereas surface tension need not be affecting waves on the bottom, it has to be influencing waves on the top since it is the only restoring force there. Therefore it appears that the character of the transition process should vary depending on the portion of the tube circumference considered. Long crested two-dimensional roll waves of the type obtained in the experiments reported in this paper would not be expected. In fact photographs presented by Kinney, et al. indicate them to be short crested.

TABLE 4. SUMMARY OF CONDITIONS DURING A RUN TO DETERMINE ROLL-WAVE TRANSITION

Liquid—water $W = 0.229$ lb./sec.
$\rho = 62.4$ lb./cu. ft.

N_{Re}	N_{Re_g}	$\frac{HR}{V^2}$	N_{Fr} (exp.)	N_{Fr} (calc.) for instability
338	4,080	0.20	0.548	1.32
338	6,120	0.19	0.720	1.43
338	7,630	0.19	0.877	1.52
338	9,050	0.151	1.13	1.72
328	10,700	0.128	1.43	1.88
328	12,100	0.127	1.76	1.97
325	13,300*	0.102	1.96	2.10
322	14,600	0.082	2.39	2.31
319	16,200	0.055	3.28	2.65
319	17,400	0.045	3.74	3.02
316	19,600	0.031	4.85	3.25

* Indicates the flow condition at which transition was observed.

The prediction of the influence which such waves would have on the pressure fluctuations in the gas phase is at present unknown. It is quite likely that the pressure fluctuations would be smaller than predicted by Equations (44), (55), and (56).

If one assumed that there were no circumferential flow in the liquid and that the liquid film is thin, then the two-dimensional analysis for the liquid might approximate the behavior of the liquid at some portion of the circumference of the tube. For such an analysis $\cos \beta = 0$, and $\sin \beta$ would vary from $+1$ at the bottom portion of the circumference to 0 at the sides to -1 at the top. If a portion of the bottom half of the tube were observed and if the pressure fluctuations in the gas phase were neglected, then one would expect a curve of the form of the theoretical curve for $P_g' = 0$ in Figure 4. The magnitude of gas Reynolds numbers could be different from those represented in Figure 4 depending on the value of $\sin \beta$ and the tube diameter. At very high gas Reynolds numbers the curve steepens considerably and shows small change in the liquid Reynolds number for transition for relatively large changes in the gas Reynolds number. The stability curve would be displaced downward and to the left for a liquid of higher viscosity. Therefore if observations were made over a fixed gas Reynolds number range, the fluid of high viscosity would show transition at a lower liquid Reynolds number.

SUMMARY

Roll-wave disturbances of the type noted in the free flow of a liquid in an inclined channel can be obtained in the concurrent flow of a gas and a liquid. These disturbances might be considered as surges in the volumetric flow which arise because of a natural instability in the flow. An approximate theory proposed by Jeffreys to predict the transition to roll waves has been extended to include gas-liquid flow and more general flows in the liquid. A comparison of this theory with an exact solution for free flow down a vertical plane gives satisfactory agreement. The extension of the theory of Jeffreys also is in good agreement with measured transitions for the concurrent flow of air and water, water-glycerine solutions, and water-butanol solutions, for which the transition occurs over a wavy surface. The addition of a surface active agent to the water completely dampens out this wavy structure but does not eliminate the transition to roll waves. The poor agreement of theory with transition data for solutions of surface active agents is not entirely explained. In treating the transition over a wavy surface it was assumed that the velocity distribution for turbulent flow near a wall is the same as the velocity distribution in the liquid film. This assumption has been checked by Hershman (*18*). In considering gas-liquid flows the effect of the fluctuation of the liquid surface on the gas flow was considered, and it was found that fluctuations of the pressure of the gas in phase with the maxima or minima in the liquid height are important. However these pressure fluctuations might not be affecting the transition noted in film-cooling experiments. The theory proposed by Lighthill and Whitman gives the same results as that of Jeffreys. This is discussed in another paper (*16*).

ACKNOWLEDGMENT

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NOTATION

A = function of Reynolds number for unsteady liquid flows equal to $\tau_w h / 2 \, \mu \, u_a$
b = height of the channel
$b-h$ = height of the gas phase
B = coefficient defined by Equation (57)
C = dimensionless complex wave velocity
C_R = actual dimensionless wave velocity, real part of the complex wave velocity
C_i = imaginary part of the dimensionless complex wave velocity

d = coefficient defined by Equation (44)
e = coefficient defined by Equation (44)
g = acceleration of gravity
h = instantaneous height of the liquid
$\overline{h}$ = time average height of the liquid
h' = fluctuation in the liquid height
h^+ = equal to $hu^* \rho/\mu$
H = equal to $\overline{h}/(b-\overline{h})$
n = exponent defined in Equation (57)
N_{Fr} = Froude number, $\overline{u}_a/(g\overline{h})^{1/2}$
N_{Re} = liquid Reynolds number, $\overline{h}\,\overline{u}_a \rho/\mu$
N_{Reg} = gas Reynolds number, $(b-\overline{h})\,\overline{u}_{ga}\rho_g/\mu_g$
N_{We} = Weber number, $\sigma/\overline{h}\rho\overline{u}_a^2$
p = pressure in the liquid
p_g = instantaneous pressure in the gas
$\overline{p_g}$ = time average pressure in the gas
p_g' = fluctuation in the pressure in the gas
R = density ratio, ρ_g/ρ
u = instantaneous velocity in the liquid in the x direction at a given value of y
u_a = mixed average velocity in the liquid flow equal to $\frac{1}{h}\int_0^h u dy$
$\overline{u_a}$ = time average of u_a
u_a' = fluction in the velocity u_a
u^* = friction velocity equal to $(\tau_w/\rho)^{1/2}$
u_g = instantaneous velocity in the gas in the x direction at a given value of y
u_{ga} = mixed average velocity in the gas equal to

$$\frac{1}{(b-h)}\int_h^b u_g dy$$

$\overline{u_{ga}}$ = time average of u_{ga}
u'_{ga} = fluctuation in u_{ga}
V = velocity ration $\overline{u}_g/\overline{u}_{ga}$
x = coordinate parallel to the direction of mean flow
X = dimensionless distance equal to $x/\overline{h}$
y = coordinate perpendicular to the direction of mean flow
y^+ = dimensionless distance equal to $y\,u^* \rho/\mu$

Greek Letters

α = wave number equal to $2\pi\overline{h}/\lambda$
β = angle which the channel makes with a line in the direction of gravity
γ = ratio of shear stresses τ_o/τ_s
Γ = liquid velocity profile factor equal to $\frac{1}{h\,u_a^2}\int_0^h u^2 dy$
$\Gamma_\omega, \Gamma_\eta$ = coefficients defined by Equation (41)
Γ_g = gas velocity profile factor equal to

$$\frac{1}{(b-h)u^2_{ga}}\int_h^b u_g^2 dy$$

δ = amplitude coefficient in Equation (12)
Δ = amplitude coefficient in Equation (13)
Δ_g = amplitude coefficient in Equation (50)
η = dimensionless fluctuating height equal to $h'/\overline{h}$
θ = time
Θ = dimensionless time equal to $\overline{h}\theta/\overline{u}_a$
λ = wave length
μ = viscosity of the liquid
μ_g = viscosity of the gas
ρ = density of the liquid
ρ_g = density of the gas
σ = surface tension
τ_s = instantaneous stress at the interface
$\overline{\tau_s}$ = time average shear stress at the interface
τ_s' = fluctuation in shear stress at the interface
$\overline{\tau_o}$ = average shear stress at the top wall of the channel
τ_w = instantaneous shear stress at the bottom wall of the channel
$\overline{\tau_w}$ = time average shear stress at the bottom wall of the channel
τ_w' = fluctuation in the shear stress at the bottom wall of the channel
ω = dimensionless velocity fluctuation in the liquid equal to $u_a'/\overline{u}_a$
ω_g = dimensionless velocity fluctuation in the gas equal to $u'_{ga}/\overline{u}_a$

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